The Statistical Performance of the MUSIC and the Minimum-Norm Algorithms in Resolving Plane Waves in Noise

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Abstract—This paper presents an asymptotic statistical analysis of the null-spectra of two eigen-assisted methods, MUSIC [1] and Minimum-Norm [2], for resolving independent closely spaced plane waves in noise. Particular attention is paid to the average deviation of the null-spectra from zero at the true angles of arrival for the plane waves. These deviations are expressed as functions of signal-to-noise ratios, number of array elements, angular separation of emitters, and the number of snapshots. In the case of MUSIC, an approximate expression is derived for the resolution threshold of two plane waves with equal power in noise. This result is validated by Monte Carlo simulations.

I. INTRODUCTION

EIGENDECOMPOSITION-based methods have found new prominence in array spectral analysis of plane signals received in noise. Examples of recent reported work in this area include those of Schmidt [1], Kumaresan and Tufts [2], Johnson and DeGraff [3], Böhme [4], and Bienvenu [6] among others. A more extensive list of research on this subject can be found in the above mentioned references. These references also contain detailed explanations of the philosophies behind the various eigenanalysis-based techniques. The aim of this paper is, therefore, neither to introduce a new array spectral estimator nor to review, in any detail, the known ones. Rather, this paper presents a statistical analysis of two of the more popular methods as reported in [1] and [2], namely, the MUSIC and the Minimum-Norm (Min-Norm) methods, with the aim of determining their resolving properties.

A common feature of the two methods discussed here is the decomposition of an estimate of the received signal covariance matrix into orthogonal "signal" and "noise" subspaces and formulating the direction-of-arrival estimator in one or the other subspace. For these methods, as well as other spectrum estimation approaches to this parameter estimation problem, the directions of arrival (DOA) are given by the positions of the spectral peaks or alternatively by locations of the nulls of the inverse of the spectrum (called the null-spectrum here). Thus, sources are "resolved" if the estimated null-spectrum contains minima (nulls) at or in the immediate neighborhoods of the true directions of arrival.

When the exact ensemble spatial covariance matrix is used, MUSIC and Min-Norm result in unbiased values (i.e., zero) for the null-spectrum of uncorrelated plane waves at the true DOA irrespective of the signal-to-noise ratios and angular separations of the sources. This is in contrast to schemes such as the autoregressive (AR) method which produces a biased null-spectrum that depends on the signal-to-noise ratios even when the exact covariance matrix is used. For a given angular separation of sources and AR model order, there is a signal-to-noise ratio below which the null-spectrum, at an angle away from (usually between) those of two close sources, is smaller than at either of the two true directions of arrival resulting in failure to "resolve" the two plane waves.

When the narrow-band spatial covariance matrix is estimated from a finite number of independent snapshots, the eigen-assisted methods also exhibit deviations from zero in their null-spectra at the true angles, resulting in a loss of resolution. This deviation is due to the statistical sampling perturbation of the signal and noise subspaces. This perturbation depends on the signal-to-noise ratios, signal parameters, and array specifications, which together determine the resolving capability of the estimation method used. In this paper we will examine the finite-sample bias in the null-spectra of the two eigen-assisted methods mentioned earlier.

The paper is organized as follows. First, the signal, noise, and array models are formulated and the MUSIC and Min-Norm methods are presented. Next, first-order approximations to the mean and variance of the MUSIC null-spectrum are derived. The expression for the MUSIC bias is used to obtain a resolution threshold for two uncorrelated, equipowered plane waves in white noise. This threshold is compared to the threshold signal-to-noise ratios obtained by Monte Carlo simulations. Finally, a first-order approximation for the mean of the Min-Norm null-
spectrum is derived. One- and two-signal-in-noise models are used to assess the dependence of the spectral bias on the various measurement parameters. An important issue that is not addressed here is the determination of the number of sources. Throughout the paper it will be assumed that the number of sources are known.

II. Formulation of the Methods

Assume \( M \) incoherent plane waves are incident on a linear equispaced array of \( L \) sensors. For the \( k \)th observation period (snapshot), the spatial samples of the signal plus noise are given by

\[
X_k = \left[ \sum_{i=1}^{M} p_i^{(k)} e^{j\omega_i}, \ldots, \sum_{i=1}^{M} p_i^{(k)} e^{j(L-1)\omega_i} \right] + N_k^T
\]

where for plane waves at a temporal frequency of \( f \) Hz, \( \omega_i = 2\pi D f / C \sin \theta_i \), \( C \) is the velocity of propagation, \( D \) is the element spacing, and \( \theta_i \) is the angle of incidence, with respect to broadside, for the \( i \)th wave. \( N_k \) is assumed to be a complex, zero-mean, circular Gaussian vector with orthogonal elements, and \( p_i^{(k)} \) are the complex amplitudes of the plane waves with \( |p_i^{(k)}|^2 = P_i \). These amplitudes are assumed to be jointly circular Gaussian and jointly independent and independent of \( N_k \).

Throughout the paper we will use the superscript "\(^\dagger\)" and \( E[\cdot] \) interchangeably to denote statistical expectation. The observation covariance matrix \( R \), and its estimate \( \hat{R} \) are given by

\[
R = E[X_k X_k^H] = \sum_{i=1}^{M} P_i R_{S_i} + \sigma_n^2 I
\]

where \( H \) denotes Hermitian transpose, \( \sigma_n^2 \) is the element noise variance, and

\[
R_{S_i} = \begin{bmatrix}
1 & e^{-j\omega_i} & \cdots & e^{-j(L-1)\omega_i} \\
e^{j\omega_i} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
e^{j(L-1)\omega_i} & \cdots & \cdots & 1
\end{bmatrix}
\]

\[
\hat{R} = \frac{1}{N} \sum_{k=1}^{N} X_k X_k^H.
\]

\( \hat{R} \) is the statistic on which the angular spectral estimates discussed in this paper are based. In the following, spectral models of interest are first reviewed. These techniques are based on an eigendecomposition of the covariance matrix into signal and noise subspaces followed by an associated spectrum, based on the signal-space or the noise-space information of the form \( D(\omega) \) and \( \hat{D}(\omega) \), its estimate for the algorithms of interest, since these functions are substantially easier to analyze than their inverses. Throughout this presentation "\(^\dagger\)" will denote the estimate of the quantity over which it appears. This estimate, in turn, is a result of using \( \hat{R} \) in place of \( R \).

We begin by writing the eigendecomposition of the covariance matrix \( R \) as

\[
R = \sum_{i=1}^{L} \lambda_i S_i S_i^H
\]

where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M > \lambda_{M+1} = \cdots = \lambda_L = \sigma_n^2 \) are the eigenvalues of \( R \), and \( S_i \) are its orthonormal eigenvectors. \( \lambda_i(S_i), i \leq M \) are the signal-space eigenvalues (eigenvectors) and the remaining ones are the noise-space eigenvalues (eigenvectors). The null-spectra for the different methods are now examined.

A. MUSIC [1]

Under the plane wave model, \( D(\omega) \) for the MUSIC method is given alternatively in terms of noise- and signal-space quantities by

\[
D(\omega) = V^H(\omega) \left( \sum_{i=M+1}^{L} S_i S_i^H \right) V(\omega)
\]

and

\[
D(\omega) = 1 - V^H(\omega) \left( \sum_{i=1}^{M} S_i S_i^H \right) V(\omega)
\]

where the steering vector \( V(\omega) \) is given by

\[
V^T(\omega) = \frac{1}{\sqrt{L}} [1, e^{j\omega_1}, \cdots, e^{j(L-1)\omega_1}].
\]

When \( V(\omega) \) coincides with a signal direction vector at angular frequency \( \omega_i \), \( D(\omega_i) = 0 \) as desired.

B. Minimum-Norm [2]

This technique finds the vector \( A \) with a unit first element which is entirely in the noise-space and has the minimum Euclidean norm. The null-spectrum is given by

\[
D(\omega) = |V^H(\omega) A|^2
\]

A particularly useful expression for \( A \) is given in [2]. We reproduce this formulation below and use it in the subsequent analysis of the Min-Norm technique. Let \( E \) be constructed as

\[
E = [S_1, S_2, \cdots, S_M]
\]

\[
= \begin{bmatrix}
g^T \\
\vdots \\
E'
\end{bmatrix}
\]

where \( g^T = [S(1), \cdots, S_M(1)] \). Then

\[
A^T = [1, -g^H E'\dagger (1 - g^H g)].
\]

It can be shown that in this case, as well, \( D(\omega_i) = 0, i = 1, \cdots, M \), as desired.
In the following, we introduce sampling errors in the estimation of $R$ resulting in statistical errors in $\hat{D}(\omega)$. Some statistics of $\hat{D}(\omega)$ are then obtained and related to the resolving capabilities of the two methods.

### III. First- and Second-Order Moments of the MUSIC Null-Spectrum

In this section, approximate statistical behavior of the MUSIC method is examined. Unfortunately, a theoretical analysis of the spectral estimators leading to a direct measure of resolution is extremely difficult at best. However, statistical bias and variance of $b(\omega)$, especially in the neighborhood of $\omega_i$, can be interpreted as indicators of the resolving capabilities of these techniques. Thus, in this section, we evaluate $E[b(\omega)]$ and $\text{var}[b(\omega)]$ for MUSIC and relate them to the angular separation and dynamic range of the sources, array signal-to-noise ratios, and number of snapshots. We close the section by deriving an expression for the resolution threshold of MUSIC in the case of two equipowered sources.

In the following analysis, we make use of the asymptotic statistics for the eigenvalues and eigenvectors of the sample covariance matrix $\hat{R}$ of a complex Gaussian process as derived partially in [5] and partially in Appendix A. Asymptotic is in the sense of large $N$ and our analysis is based on the first-order perturbation of $D(\omega)$ and $D(\omega)^2$ in the neighborhoods of the signal angles. One- and two-signal examples are specifically treated to obtain illustrative results. We begin by summarizing the first- and second-order statistics of the estimation errors in $f_i$ and $S_i$ as given in Appendix A. Let $\hat{S}_i = S_i + \eta_i$ and $\hat{\lambda}_i = \lambda_i + \beta_i$, where $\lambda_i$ is the $i$th distinct eigenvalue of $R$. Then $\eta_i$ and $\beta_i$ have the following asymptotic properties:

$$E[\beta_i \beta_j] = \frac{\lambda_i^2}{N} \delta_{ij}. \quad (11)$$

Using (A.15) and (A.13)

$$E[\eta_i \eta_j^H] = \frac{\lambda_i}{N} \sum_{k=1}^{\infty} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2} S_k \bar{S}_k \delta_{ij}, \quad (12)$$

from (A.16)

$$E[\eta_i \eta_j^T] = -\frac{\lambda_i \lambda_j}{N(\lambda_i - \lambda_j)^2} S_j \bar{S}_j^T (1 - \delta_{ij}), \quad (13)$$

and (A.13) gives

$$E[\eta_i] = -\frac{\lambda_i}{2N} \sum_{k=1}^{\infty} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2} S_k \delta_{i} \triangleq \alpha_i S_i. \quad (14)$$

In the above, $\delta_{ij}$ is the Kronecker delta and the approximations are $o(N^{-1})$.

It is worth noting that (14) is consistent with (12) and the normalization constraint on $|\hat{S}_i|$ and $|S_i|$ as can be seen in the following. The constraints result in

$$|\hat{S}_i|^2 = S_i^H S_i + S_i^H \eta_i + \eta_i^H S_i + \eta_i^H \eta_i = 1 \quad (15)$$

which gives

$$S_i^H \eta_i + \eta_i^H S_i = -\eta_i^H \eta_i. \quad (16)$$

Taking the expectation of (16) and retaining order $N^{-1}$ terms results in

$$2 \text{Re} [S_i^H \eta_i] = -\frac{\lambda_i}{N} \sum_{k=1}^{\infty} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2} S_k^H S_k$$

$$= -\frac{\lambda_i}{N} \sum_{k=1}^{\infty} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2}. \quad (17)$$

Clearly, the expression for $\eta_i$ in (14) satisfies the constraint in (17). We now proceed to derive the first- and second-order statistics of the null-spectrum of MUSIC.

The estimated null-spectrum for this method is given by

$$\hat{D}(\omega) = 1 - V^H(\omega) \left( \sum_{i=1}^{M} \hat{S}_i \hat{S}_i^H \right) V(\omega). \quad (18)$$

The expected value of $\hat{D}(\omega)$, using the definition of $\hat{S}$, is given by

$$\bar{D}(\omega) = 1 - V^H(\omega) \left( \sum_{i=1}^{M} S_i S_i^H \right) V(\omega)$$

$$- V^H(\omega) E \left[ \sum_{i=1}^{M} \eta_i \eta_i^H \right] V(\omega)$$

$$- 2 \text{Re} \left[ V^H(\omega) \left( \sum_{i=1}^{M} S_i \eta_i^H \right) V(\omega) \right]. \quad (19)$$

Substituting for the expectations in (19) gives

$$\bar{D}(\omega) = D(\omega) - V^H(\omega)$$

$$\left[ \sum_{i=1}^{M} \sum_{j=1}^{L} \frac{\lambda_i \lambda_j}{N(\lambda_i - \lambda_j)^2} (S_j S_j^H - S_i S_i^H) \right] V(\omega). \quad (20)$$

Since our primary objective is the investigation of the resolving capability for these methods, we will concentrate on the statistics of $\bar{D}(\omega)$ in the neighborhood of $\omega_i$, $i = 1, \ldots, M$. The results for $M = 1$ and $M = 2$ follow immediately, noting that $V(\omega_i) \in \text{span} \{S_i\}, i = 1, \ldots, M$ and, therefore, $V(\omega_i)$ is orthogonal to $S_i, i = M + 1, \ldots, L$. It follows that

$$M = 1: \quad \bar{D}(\omega_1) = \frac{\lambda_1 \sigma_n^2(L - 1)}{N(\lambda_1 - \sigma_n^2)}. \quad (21)$$

$$M = 2: \quad \bar{D}(\omega_2) = \sigma_n^2 V^H(\omega_2) \left[ \frac{\lambda_2(L - 2)}{(\lambda_1 - \lambda_2)} S_1 S_1^H \right. \right.$$}

$$+ \left. \frac{\lambda_1(L - 2)}{(\lambda_2 - \sigma_n^2)} S_2 S_2^H \right] V(\omega_2)/N. \quad (22)$$
The next quantity of interest is the variance of $\bar{D}(\omega)$ for which we again obtain the order of $N^{-1}$ approximation. It is convenient to define $D(\omega)$ as

$$D(\omega) = 1 - D_0(\omega)$$

where $D_0(\omega)$ is the null-complement spectrum. The variance of $D(\omega)$ is then given by

$$\text{var} [\bar{D}(\omega)] = \text{var} [D_0(\omega)].$$

Therefore, we first derive an expression for $\bar{D}_c^2(\omega)$. From the definition of $\bar{D}(\omega)$ (18), it follows that

$$\bar{D}_c^2(\omega) = V^H(\omega) \bar{T} V(\omega)$$

where

$$\bar{T}(m, n) = \sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{l=1}^{L} \sum_{k=1}^{L} \left\{ \sum_{i=1}^{M} \sum_{j=1}^{M} [S_i(k) S^*_j(n) W_i(m, l) + S^*_i(l) S_j(n) W_j(k, l) \delta_{ij} + S_i(k) S_j(m) U^*_{ij}(n, l) + S^*_i(n) S^*_j(l) U_{ij}(k, m) + S^*_i(n) S_j(m) W_j(k, l) \delta_{ij} + S_i(k) S^*_j(l) S^*_j(n) S_j(m) + 2(a_i + a_j) S_i(\xi) S^*_j(l) S_j(m) S^*_n(n)] \right\}$$

where

$$W_j(i, m) = (i, m)\text{th element of } E[\eta_j \eta^*_j]$$

and

$$U_{ij}(i, m) = (i, m)\text{th element of } E[\eta_i \eta^*_j].$$

The expression in (25) is quite unwieldy with doubtful general informational value. For one and two signals in white noise, $\bar{D}_c^2(\omega)$ can be simplified using the statistics of $\eta_i$ and the orthogonality of $V(\omega)$ and the noise eigenvectors. These results are briefly examined as follows.

1) $M = 1$:

$$\bar{T} = \sum_{k=2}^{L} S_k S^*_k \left| V^H(\omega) S_k \right|^2 \frac{\lambda_1 \lambda_k}{N(\lambda_1 - \lambda_k)^2} + \left| V^H(\omega) S_1 \right|^2 S_1 S^*_1(1 + 4a_1).$$

Therefore,

$$\bar{D}_c^2(\omega) = 1 + 4a_1 = 1 - \frac{2\lambda_1 \lambda_2 (L - 1)}{N(\lambda_1 - \lambda_2)^2}$$

resulting in

$$\text{var} [\bar{D}(\omega)] = \bar{D}_c^2(\omega) + 2\bar{D}(\omega) - 1 - (\bar{D}(\omega))^2 \approx 0.$$  \hfill (27)

This is not surprising as the one signal case is a rather stable one.

2) $M = 2$: Using the second-order statistics (12) and (13) and simplifying the result at $\omega_k$, $k = 1, 2$, leads to

$$\bar{T} = \sum_{j=1}^{2} \sum_{i=1}^{2} \left\{ V^H(\omega_k) S_i S^*_i V(\omega_k) \right\} S_j S^*_j$$

$$+ \left[ \frac{2\lambda_1 \lambda_2}{(N(\lambda_1 - \lambda_2)^2 + 1) + 2(a_i + a_j)} \right] V^H(\omega_k)$$

$$S_i S^*_i V(\omega_k) S_j S^*_j.$$

Therefore,

$$\bar{D}_c^2(\omega_k) = \frac{1}{2} \sum_{j=1}^{2} \left[ | V^H(\omega_k) S_j |^2 - | V^H(\omega_k) S_1 |^2 \right]$$

$$- \left( \frac{2}{\sum_{i=1}^{2} (\lambda_i - \sigma_n^2) \lambda_i} \right) | V^H(\omega_k) S_1 S^*_1 V(\omega_k) |^2.$$

\hfill (29)
We now evaluate \( \text{var} [\hat{\Delta}(\omega)] \) and \( \overline{\Delta}(\omega) \) in terms of the signal, noise, and array parameters of interest. The deviation of \( \overline{\Delta}(\omega) \) from zero, the null-spectral bias, signifies a loss in resolution of the spectral estimator. We first give approximate expressions for this bias, for later comparison to Min-Norm. Subsequent analysis approximates the resolution threshold for two equipowered, closely spaced uncorrelated sources as that signal-to-noise ratio at which

\[
\overline{\Delta}(\omega), \quad k = 1, 2, \quad \text{is equal to} \quad \overline{\Delta}\left(\frac{\omega_1 + \omega_2}{2}\right).
\]

For one signal in noise it was shown in (27) that \( \text{var} [\hat{\Delta}(\omega_1)] = 0. \) \( \overline{\Delta}(\omega_1) \) for \( LP_1 \gg \sigma_n^2 \) is given by

\[
\overline{\Delta}(\omega_1) = \frac{(L - 1)\sigma_n^2}{NLP_1}. \tag{31}
\]

Thus, the bias at \( \omega_1 \) is directly proportional to the noise-to-signal ratio and inversely proportional to the number of snapshots as expected.

For closely spaced equal-power signals, using (B-7) and (B-8) from Appendix B in (30) and retaining the two dominant terms gives

\[
\text{var} [\hat{\Delta}(\omega)] = \frac{3(L - 2)\Delta^2 (1 + 2\text{ASNR})}{160N(\text{ASNR})^2} \tag{32}
\]

where ASNR = \( LP/\sigma_n^2 \) is the array signal-to-noise ratio and \( \Delta = L(\omega_1 - \omega_2)/2\sqrt{3} \) is proportional to the normalized difference between sines of the DOA. It is clear that this variance decreases as \( (\text{ASNR})^{-2} \) for ASNR \( \ll 1/2 \) and as \( (\text{ASNR})^{-1} \) for ASNR \( \gg 1/2 \). For the latter region (the practical one) and large \( L \), the variance is actually determined by the inverse of the elemental signal-to-noise ratio, \( P_1/\sigma_n^2 \).

Again using the results of Appendix B in (22) and retaining the largest two terms, we obtain approximate expressions for \( \overline{\Delta}(\omega) \). For equal power sources, we get

\[
\overline{\Delta}(\omega) = \frac{(L - 2)}{N}\left[\frac{1}{2(\text{ASNR})} + \frac{1}{(\text{ASNR})^2\Delta^2}\right]. \tag{33}
\]

It is clear from (32) and (33) that, for reasonable array parameters, the bias in \( \overline{\Delta}(\omega) \) is substantially larger than its standard deviation (SD). For example, for \( L = 10, \Delta^2 = 0.003, \) and \( N = 100, \overline{\Delta}(\omega)/\text{SD} [\hat{\Delta}(\omega)] = 10^4 \). Thus, we expect that the resolution threshold is mostly determined by the behavior of \( \overline{\Delta}(\omega) \).

For two sources with unequal powers as given also in Appendix B, we obtain the following approximations to \( \overline{\Delta}(\omega) \):

\[
\overline{\Delta}(\omega_1) = \frac{(L - 2)}{N\Delta^2 (\text{ASNR})^2} \left[\frac{1}{L} + \frac{\Delta^2}{\Delta^2 + \frac{3}{2}\delta} \right] + \frac{1}{\Delta^2} \left(\frac{\Delta^2}{L} + \frac{3}{2}\delta \Delta^2\right) \tag{34}
\]

where ASNR in this case refers to the weaker source array signal-to-noise ratio and \( \Delta^2 \ll \delta = P_2/P_1 \ll 1 \). A similar expression may be obtained for \( \overline{\Delta}(\omega_2) \).

The above expressions for \( \overline{\Delta}(\omega) \) indicate the general behavior of the null-spectrum bias as a function of source and array parameters. In order to obtain a quantitative measure of the resolution threshold for two closely spaced equipowered sources, we now introduce a plausible, nonprobabilistic approach based on \( \overline{\Delta}(\omega) \). We propose that the signal-to-noise ratio at which \( \overline{\Delta}(\omega_1) = \overline{\Delta}(\omega_2) = \overline{\Delta}(\omega_m) \), where \( \omega_m = (\omega_1 + \omega_2)/2 \), is approximately this threshold. The reasons for this conjecture are as follows.

Resolution is achieved when \( \overline{\Delta}(\omega_1) \) and \( \overline{\Delta}(\omega_2) \) are both less than \( \overline{\Delta}(\omega_m) \). When the above equality is true, the probability of resolution ranges from approximately 0.33, for the case of totally independent variations of \( \overline{\Delta}(\omega_1) \), \( \overline{\Delta}(\omega_2) \), and \( \overline{\Delta}(\omega_m) \), to nearly 0.5 for the situation when \( \overline{\Delta}(\omega_1) \) and \( \overline{\Delta}(\omega_2) \) are maximally correlated.

Fig. 1 shows the resolution probabilities for MUSIC and Min-Norm of a typical array obtained by Monte Carlo simulations. It is clear that the ASNR for probabilities of 0.33 and 0.5 are within 1-2 dB of each other. Thus, the proposed method should give the approximate ASNR for the 0.3-0.5 probability of resolution threshold region. Note also that equating the averages of \( \overline{\Delta}(\omega_1), \overline{\Delta}(\omega_2), \) and \( \overline{\Delta}(\omega_m) \) is justified due to the small standard deviation of \( \overline{\Delta}(\omega) \) relative to its mean. We again substitute the expressions from Appendix B in (22) and retain all terms to obtain \( \overline{\Delta}(\omega_1) \) and \( \overline{\Delta}(\omega_2) \), and in (20) to find \( \overline{\Delta}(\omega_m) \). Threshold occurs at the ASNR for which \( \overline{\Delta}(\omega_m) = \overline{\Delta}(\omega_1) \) with the resolution probability increasing as \( \overline{\Delta}(\omega_1) \) becomes increasingly less than \( \overline{\Delta}(\omega_m) \). Let the threshold ASNR be denoted by \( \xi_7 \). Then

\footnote{In each simulation trial, the two sources were considered resolved if the spectral output of the algorithm under test had two peaks, each within a beamwidth of the respective true source location, and also the power estimate \( m \) for each source was not more than 10 dB above the true power of the source.}
The evaluation of the moments of the null-spectrum for this method is in general considerably more complex than the previous one. In the following, an approximate expression for $\hat{D}(\omega_k)$ is derived. The approach is again based on a perturbation of the exact quantities with small random disturbances as discussed before. We use (10) in conjunction with the estimate $\hat{A}$ where

$$
\hat{A}^T = [1, -g^H E^T/(1 - g^H g)].
$$

The null-spectral estimate is then

$$
\hat{D}(\omega) = |V^H(\omega)\hat{A}|^2 = V^H(\omega)\hat{A}^H A^H V(\omega).
$$

Let $\hat{A} = A + \alpha \cdot \hat{D}(\omega)$ is then given by

$$
\hat{D}(\omega) = D(\omega) + 2Re\left[V^H(\omega) \alpha A^H V(\omega)\right]
$$

$$
+ V(\omega)^H \alpha \alpha^H V(\omega).
$$

It is of interest again to consider the behavior of $\hat{D}(\omega)$ at signal frequencies $\omega_k$. Using the fact that $D(\omega_k) = |A^H V(\omega_k)|^2 = 0$, $\hat{D}(\omega_k)$ reduces to

$$
\hat{D}(\omega_k) = V^H(\omega_k) \alpha \alpha^H V(\omega_k).
$$

Thus, we have to find an expression for the correlation matrix of $\alpha$. Denote the random perturbation in $E'$ and $g$ by $\epsilon'$ and $\gamma$, respectively. Then

$$
\hat{E}' = E' + \epsilon' \quad \text{and} \quad \hat{g} = g + \gamma.
$$

We now find an approximation to $\alpha$ and then derive an expression for $\alpha \alpha^H$. Note that the first- and second-order statistics of $\epsilon'$ and $\gamma$ are those of the appropriate elements of the set $\{\eta_i\}, i = 1, \cdots, M$, [see (9)]. By definition we have

$$
\alpha = \begin{bmatrix} 0 \\ E'g^* - E'\hat{g}^* \\ 1 - g^H g - 1 - \hat{g}^H \hat{g} \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha' \end{bmatrix}
$$

Therefore, $\hat{D}(\omega_k) = V^H(\omega_k) \alpha' \alpha'^H V(\omega_k)$. $V'$ is a vector formed from the second to the $L$th elements of $V$. We now
develop an approximate expression for $\alpha'$ in terms of the lowest order perturbations as follows:

$$
\frac{1}{1 - g^H g} \approx \frac{1 + \rho}{1 - g^H g}
$$

(42)

where

$$
\rho = \frac{2 \text{Re} \left[ g^H \gamma \right]}{1 - g^H g}
$$

(43)

and

$$
\alpha' = \frac{-1}{1 - g^H g} \left( \rho g^* \gamma + E' \gamma * + H + \epsilon' g^* \right).
$$

(44)

Consequently,

$$
E[\alpha' \alpha'^H] = \frac{1}{[1 - g^H g]} \left[ \rho^2 E' g^* g^T E'^H
+ E' \gamma * \gamma ^T E'^H
+ \epsilon' g^* g^T \epsilon'^H + Q + Q'^H \right]
$$

(45)

where

$$
Q = E' g^* \rho \gamma ^T E'^H + E' g^* \epsilon ^T \epsilon ^H
+ E' \gamma * \epsilon ^T \epsilon ^H.
$$

(46)

The quantities under the expectation operations are evaluated in Appendix C. In the following, these results are used to obtain $\overline{D}(\omega)$ for $M = 1$ and $M = 2$. Unfortunately, $\overline{D}(\omega)$ is analytically intractable for Min-Norm. Therefore, no expression for a threshold ASNR is derived for this method.

For $M = 1$ we have $g = S_1(1) = 1/\sqrt{L}$ and

$$
\frac{1}{(1 - \frac{1}{L})^2} V^H(\omega_1) E' E'^H V(\omega_1) = 1.
$$

(47)

We now make a further assumption that $L \gg 1$. This assumption is not crucial but allows for a more manageable expression for $\overline{D}(\omega)$. Substituting the various quantities from Appendix C into (45) and noting the dominance of the second term in (45) results in

$$
\overline{D}(\omega) = V^H(\omega) E' \gamma ^T E'^H V(\omega) \cdot \frac{1}{(1 - \frac{1}{L})^2}
$$

$$
= \frac{\sigma_n^2}{L P_1 \left( 1 - \frac{1}{L} \right)} \left| V^H(\omega) E' \right|^2 \sum_{k=2}^{L} |S_k(1)|^2.
$$

(48)

The sum in (48) can be found using the orthonormality of $\{S_k\}$ as

$$
\sum_{k=2}^{L} |S_k(1)|^2 = 1 - \frac{1}{L}.
$$

Then

$$
\overline{D}(\omega) = \frac{\sigma_n^2}{L P_1 \left( 1 - \frac{1}{L} \right)} \left( 1 - \frac{1}{L} \right).
$$

(49)

For $M = 2$ we consider the case of closely spaced sources with equal powers. Again, the second term in (45) dominates, resulting in

$$
\overline{D}(\omega) = \frac{1}{1 - \left( |S_1(1)|^2 + |S_2(1)|^2 \right)} \cdot \{ \Gamma_1 |V^H(\omega) F_1|^2 + \Gamma_2 |V^H(\omega) F_2|^2 \}
$$

(50)

where $\Gamma_i = \left[ \gamma^* \gamma \right]_i$ and $F_i$ is defined in Appendix C as the vector of the second to $L$th elements of $S_i$. Following the same procedure as for $M = 1$, we obtain

$$
\overline{D}(\omega) = \lambda_1 \sigma_n^2 \left( 1 - \frac{|S_1(1)|^2}{\lambda_1 - \sigma_n^2} \right)
$$

$$
+ \lambda_i \sigma_n^2 \left( \frac{\lambda_2 - \lambda_i}{\lambda_i - \sigma_n^2} \right) |S_{3-i}(1)|^2.
$$

(51)

Substituting the values from Appendix B, we obtain

$$
\overline{D}(\omega) = \frac{1}{1 - \frac{4}{L} N} \left[ \frac{1}{(\text{ASNR})} + \frac{1}{(\text{ASNR})^2 \Delta^2} \right].
$$

(52)

The above results on $\overline{D}(\omega)$ are somewhat optimistic for Min-Norm. There are other, smaller terms in (45) that contribute to the null-spectral bias. Nevertheless, the dominant makeup of this quantity does indicate a smaller bias (by nearly a factor of $L$) as compared to MUSIC. This might be expected to carry over to the relative resolution thresholds for the two methods, making the Min-Norm threshold lower than that of MUSIC. This difference in the resolution thresholds has been verified in numerous simulations, an example of which is shown in Fig. 1.

V. CONCLUSION

This paper presented an asymptotic evaluation of the resolving capability of two eigen-assisted spectral domain estimators of the directions of arrival of closely spaced, narrow-band plane waves. The mean and variance and the mean, respectively, of the null-spectra of the MUSIC and Minimum-Norm algorithms, including $O(N^{-1})$ errors, were derived. These were approximately related to the resolving power of the two methods and their dependence on such parameters as the relative angular separation of emitters, number of sensors, number of snapshots, and signal-to-noise ratios were delineated. For the MUSIC algorithm, an expression for a plausible detection threshold

3 Of course, we have not analyzed $\overline{D}(\omega_1)$ here, nor have we shown that the standard deviation of the Min-Norm null-spectrum is small as we showed for MUSIC. Hence, the analytical comparison of the methods is conjectural.
was derived that showed close agreement with results from Monte Carlo simulations. Our results indicate a smaller bias in the Min-Norm null-spectrum, at a source angle, compared to the MUSIC null-spectrum. This suggests (but does not imply) a resolution threshold which is at a lower signal-to-noise ratio for Min-Norm, a fact supported by simulation results. Above and below these thresholds simulations have shown the two methods to behave similarly in terms of resolution probability, bias, and variance of the estimated angles.

APPENDIX A

In this appendix, we derive the \( o(N^{-1}) \) approximations to the first- and second-order moments of \( \hat{\lambda}_i \) and \( \hat{S}_i \) based on the perturbation formulations in [7] and the statistical results in [5]. It is understood that the statistics that are given in the following are for the distinct eigenvalues (signal-space eigenvalues) and their associated eigenvectors. \( \hat{\lambda}_i \) are asymptotically normal with \( \lambda_i \) and \( \hat{S}_i \), \( i, j = 1, \ldots, L \), asymptotically independent. Furthermore [5],

\[
\overline{\hat{\lambda}_i} = \lambda_i + o(N^{-1}) \quad (A.1)
\]

\[
\text{cov} (\hat{\lambda}_i, \hat{\lambda}_j) \approx \delta_{ij} \lambda_i^2 / N. \quad (A.2)
\]

No explicit expression for \( \hat{S}_i \) is given in [5]. Moreover, all the existing derivations of the statistics of the eigenvector estimates neglect the numerical normalization of \( \hat{S}_i \). In the following, we derive the \( o(N^{-1}) \) approximation to the first- and second-order moments of \( \hat{S}_i \) assuming that \( |\hat{S}_i| = 1 \).

Following Wilkinson's approach [7, p. 68], we define \( \hat{R} \) in terms of a random perturbation to \( R \) with a perturbation factor \( p \), \( 0 < p \ll 1 \). Thus,

\[
\hat{R} = R + (\hat{R} - R) = R + pB \quad (A.3)
\]

where \( B \) is a Hermitian, zero-mean random matrix with elements that are asymptotically jointly Gaussian, obviously defined by (A.3). We will need the following result later. Let \( A_i \), \( i = 1, 2, 3, 4 \), be four complex vectors with appropriate dimensions. The following expression results (see [5, p. 114]):

\[
E[(A_i^H B A_i^H) (A_j^H B A_j^H)] = \frac{(A_i^H R A_i^H) (A_j^H R A_j^H)}{N \lambda_i \lambda_j}. \quad (A.4)
\]

Let \( \hat{S}_i \) denote the unnormalized eigenvector given in a perturbation expansion as in [7] by

\[
\hat{S}_i = S_i + \sum_{j \neq i}^{\infty} \left( \sum_{k=1}^{L} t_{jk}^{(i)} p^k \right) S_j \quad (A.5)
\]

where \( t_{kj}^{(i)} \), \( k = 1, 2, \ldots \), are the coefficients of the perturbation expansion of \( \hat{S}_i \) along \( S_j \). The following steps develop the lowest order perturbation expression for the normalized estimate of the \( i \)th eigenvector \( \hat{S}_i \). Using the orthonormality of \( S_i \) and keeping the term with the lowest order of \( p \), we have

\[
|\hat{S}_i|^2 \approx 1 + \sum_{j \neq i}^{L} |t_{ij}^{(i)}|^2 p^2 \quad (A.6)
\]

\[
|\hat{S}_i|^{-1} = 1 - \frac{1}{2} \sum_{j \neq i}^{L} |t_{ij}^{(i)}|^2 p^2 \quad (A.7)
\]

and

\[
\hat{S}_i = \hat{S}_i |\hat{S}_i|^{-1} \approx \left( 1 - \frac{1}{2} \sum_{j \neq i}^{L} |t_{ij}^{(i)}|^2 p^2 \right) S_i
\]

\[
+ \frac{L}{2} \sum_{j \neq i}^{L} t_{ij}^{(i)} p S_j + \sum_{j = 1}^{L} t_{ij}^{(i)} p S_j. \quad (A.8)
\]

The expressions for \( t_{ij}^{(i)} \) are obtained by substitution of (A.8) into the perturbed equation for the eigenvalue problem. Following [7], we have

\[
t_{ij}^{(i)} = \frac{S_i^H B S_j}{\lambda_j - \lambda_i}, \quad j \neq i \quad (A.9)
\]

and

\[
t_{ij}^{(2)} = \frac{(S_i^H B S_j)(S_j^H B S_i)}{\lambda_j - \lambda_i^2}
\]

\[
- \frac{L}{2} \left( \sum_{k=1}^{L} (S_i^H B S_j)(S_j^H B S_i) \right). \quad (A.10)
\]

The expected value of \( \hat{S}_i \) in (A.8) only involves \( E[|t_{ij}^{(i)}|^2] \) and \( E[t_{ij}^{(2)}] \), as \( E[t_{ij}^{(i)}] = 0 \) from (A.9). Using (A.4) and the fact that \( S_i \) are eigenvectors of \( R \) (i.e., \( S_i^H R S_j = \delta_{ij} \lambda_i \)), it follows that

\[
E[|t_{ij}^{(i)}|^2] = \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2 N \lambda_i}, \quad i \neq j \quad (A.11)
\]

and

\[
E[t_{ij}^{(2)}] = 0. \quad (A.12)
\]

Thus,

\[
E[\hat{S}_i] = S_i - \frac{1}{2} \sum_{j=1}^{L} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} N \lambda_i S_j. \quad (A.13)
\]

The approximation of order \( N^{-1} \) to the central second-order moments of \( \hat{S}_i \) are the same as those given by Brillinger ([5, p. 454]) for \( \{\hat{S}_i\}_i \). This can be seen from the following. Using (A.7) in the definition of \( \hat{S}_i \), we obtain

\[
\text{cov} (\hat{S}_i, \hat{S}_j) = \text{cov} (\hat{S}_i, \hat{S}_j)
\]

\[
- \frac{1}{2} \left\{ \text{cov} \left( \sum_{k \neq i}^{L} |t_{ik}^{(i)}|^2 p^2 \hat{S}_i, \hat{S}_i \right) \right\}
\]

\[
+ \text{cov} \left( \hat{S}_i, \sum_{k \neq i}^{L} |t_{ik}^{(i)}|^2 p^2 \hat{S}_j \right)
\]

\[
+ \frac{L}{4} \text{cov} \left( \sum_{k \neq i}^{L} |t_{ik}^{(i)}|^2 \hat{S}_i, \sum_{k \neq j}^{L} |t_{ik}^{(i)}|^2 \hat{S}_j \right) p^4.
\]
Careful examination of the last three terms in the above equation shows that they involve third or higher order moments of elements of the zero-mean complex Gaussian matrix \( B \). Thus, these terms contribute zero (for odd orders) or quantities of order \( O(N^{-2}) \). It follows that

\[
\text{cov} (\hat{S}_i, \hat{S}_j) = \text{cov} (\hat{S}_i, \hat{S}_j) + o(N^{-2}). \tag{A.14}
\]

Thus ([5, p. 343]),

\[
\text{cov} (\hat{S}_i, \hat{S}_j) = \sum_{k=1}^{L} \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} \frac{1}{N} S_i S_k^H \delta_{ij} \tag{A.15}
\]

and

\[
\text{cov} (\hat{S}_i, \hat{S}_j^*) = -\frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \frac{1}{N} S_i S_k^H (1 - \delta_{ij}). \tag{A.16}
\]

Formulas (12) and (13) follow from (A.15) and (A.16) by neglecting terms of order \( N^{-2} \) or less.

**APPENDIX B**

This appendix develops expressions for the eigenvalues, eigenvectors, and several associated inner products that are needed for the approximate evaluation of the moments of the null-spectra for one and two signal cases. For one signal in noise \( S_1 = \mathcal{V}(\omega_1) \) and \( \lambda_1 = LP_1 + \sigma_n^2 \).

For two uncorrelated signals, define centered direction vectors by

\[
U_i = e^{-j(L-1)\omega_\pi} V(u_i), \quad i = 1, 2.
\]

We consider two cases: equal and unequal power sources. In both cases the sources will be assumed to be closely spaced with respect to the array beamwidth. Thus, let \( d \) and \( \Delta^2 = L^2 \omega^2 / 3 \). Then

\[
|\mathcal{H}(\omega_i)| \leq \frac{\lambda_i^{1/2}}{2PL} \approx 1 - \frac{\Delta^2}{4} + \frac{3}{80} \Delta^4 \quad i = 1, 2 \tag{B.7}
\]

\[
|\mathcal{H}(\omega_i)| \leq \frac{\lambda_i^{1/2}}{2PL} \approx \frac{\Delta^2}{4} - \frac{3}{80} \Delta^4 \quad i = 1, 2 \tag{B.8}
\]

and

\[
|\mathcal{H}(\omega_i)| \leq \frac{1}{80} \Delta^4 \tag{B.9}
\]

\[
|\mathcal{H}_{m^2}(\omega_i)| = 0. \tag{B.10}
\]

We next consider an extreme case for which one signal dominates the other.

2) \( \Delta^2 << (P_2/P_1) = \delta << 1 \): To obtain manageable expressions in this case, we only retain terms of the lowest possible order in \( \delta \) and \( \Delta^2 \). Expanding the square root in (B.3), we get

\[
\frac{\lambda_1}{P_1} = L(1 + \delta) \quad \text{and} \quad \frac{\lambda_2}{P_1} = \frac{L \delta \Delta^2}{1 + \delta}. \tag{B.11}
\]

Substituting (B.11) in (B.4) and carrying out the normalization of \( S_i \) the following approximate expressions develop:

\[
|\mathcal{H}(\omega_1)|^2 = (1 - \Delta^2) \left( 1 + \frac{L}{L - 1} \delta \right)^2 \tag{B.12}
\]

\[
|\mathcal{H}(\omega_2)|^2 = (1 - \Delta^2)(1 + \delta)^2 \tag{B.13}
\]

\[
|\mathcal{H}_{m^2}(\omega_1)|^2 = \left( \delta + \frac{3 \Delta^2}{2} \right)^2 \tag{B.14}
\]

\[
|\mathcal{H}_{m^2}(\omega_2)|^2 = \left( \delta + \frac{3 \Delta^2}{2} \right)^2 \tag{B.15}
\]

**APPENDIX C**

In this appendix, we derive approximate expressions for the expectations in equation (45). Again, we only keep the first-order terms.

1) Let \( [A]_{ij} \) denote the \( ij \)-th element of the matrix \( A \). Then the second-order statistics of \( \gamma \) are given by

\[
[\gamma^* \gamma]^H_{ij} = \sum_{k=1}^{L} \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} |S_k(1)|^2 \delta_{ij}. \tag{A.16}
\]

\[
[\gamma^* \gamma][\gamma^* \gamma]^H = \sum_{k=1}^{L} \frac{\lambda_i \lambda_k}{(\lambda_i - \lambda_k)^2} |S_k(1)|^2 \delta_{ij}. \tag{C.1}
\]
and

$$\frac{[\mathbf{\gamma} \mathbf{\gamma}^H]_{ij}}{\mathbf{\gamma}^H \mathbf{\gamma}^H} = \frac{-\lambda_i \lambda_j}{N(\lambda_i - \lambda_j)^2} S_j(1) S_j(1)(1 - \delta_{ij}). \quad (C.2)$$

2) $\rho^2 = \frac{2}{(1 - g^H g)^2} [g^H \mathbf{\gamma}^H g + \text{Re}(g^H \mathbf{\gamma}^H \mathbf{\gamma} g)]$.

Simplifying, we obtain

$$\rho^2 = \frac{2}{(1 - g^H g)^2} \sum_{i=1}^{L} \sum_{k=M+1}^{K} \frac{\lambda_i \lambda_k}{N(\lambda_i - \lambda_k)^2} |S_k(1)|^2 |S_j(1)|^2. \quad (C.3)$$

3) $e^H g^H e^H$ is derived by noting that $[e']_{ij} = \eta_j(i + 1)$ and $g(i) = S_j(1)$

$$e^H g^H e^H = \sum_{i=1}^{M} |S_j(1)|^2 \sum_{j \neq i}^{L} \frac{\lambda_i \lambda_j}{N(\lambda_i - \lambda_j)^2} F_i F_j^H. \quad (C.5)$$

where $F_j^H = [S_j(2), S_j(3), \ldots, S_j(L)]$.

4) $\rho \mathbf{\gamma}^H = \frac{1}{1 - g^H g} [g^H \mathbf{\gamma} \mathbf{\gamma}^H + \mathbf{\gamma}^H g \mathbf{\gamma}^H]. \quad (C.6)$

From (C.2), we get

$$[g^H \mathbf{\gamma} \mathbf{\gamma}^H]_{ij} = \frac{-\lambda_i \lambda_j}{N} \sum_{i \neq j}^{M} \frac{\lambda_j}{(\lambda_j - \lambda_i)^2} |S_j(1)|^2. \quad (C.7)$$

The second term follows from (C.1) as

$$[\mathbf{\gamma} \mathbf{\gamma}^H]_{ij} = \sum_{k=x}^{L} \frac{\lambda_j \lambda_k}{N(\lambda_j - \lambda_k)^2} N S_j(1) |S_k(1)|^2. \quad (C.8)$$

5) $\rho e^H = \frac{1}{1 - g^H g} [g^H \mathbf{\gamma} e^H + \mathbf{\gamma}^H g e^H]. \quad (C.9)$

Again, we first consider the first term in the brackets.

$$g^H \mathbf{\gamma} e^H = \sum_{i=1}^{M} S_i(1) \eta_i(1) e^H. \quad (C.10)$$

Therefore,

$$[g^H \mathbf{\gamma} e^H]_{ij} = \sum_{k=x}^{L} \frac{\lambda_j \lambda_k}{N(\lambda_j - \lambda_k)^2} N S_i(1) S_k(1)(j + 1) \quad (C.11)$$

$j = 1, \ldots, L - 1.$

The second term is given by

$$\mathbf{\gamma}^H g e^H = \sum_{i=1}^{M} S_i(1) \eta_i(1) e^H. \quad (C.12)$$

resulting in the $i$th element of the matrix as

$$[\mathbf{\gamma}^H g e^H]_{ij} = \sum_{k=x}^{L} \frac{\lambda_j \lambda_k}{N(\lambda_j - \lambda_k)^2} N S_i(1) S_k(1)(j + 1). \quad (C.13)$$

6) The final term of interest is $\mathbf{\gamma}^H g e^H$.

First, consider the $j$th element of $g e^H$

$$[g e^H]_{ij} = \sum_{k=x}^{M} S_i(1) \eta_k(j + 1). \quad (C.14)$$

Then

$$[\mathbf{\gamma}^H g e^H]_{ij} = \sum_{k=x}^{M} S_i(1) \eta_k(j + 1) - \sum_{k=x}^{M} \frac{\lambda_j \lambda_k}{N(\lambda_j - \lambda_k)^2} |S_k(1)|^2 S_k^*(j + 1). \quad (C.15)$$

ACKNOWLEDGMENT

The authors are indebted to Drs. K. D. Senne, L. Horowitz, and D. DeLong of Lincoln Laboratory for many helpful discussions during the course of this investigation.

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